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Determination of Circular and Spherical Position-Error Bounds in System Performance Analysis

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Introduction

THIS Note presents a methodology for determining circular and spherical position-error bounds for Gaussian random-error processes, such as are commonly encountered in covariance-based-navigation system performance analysis. Such analyses typically require an assessment of the achievable position accuracy in the form of a planar- or spatial-error characterization. Planar position error is conveniently characterized by establishing the radius of a circle that encompasses a prescribed percentage of all possible outcomes in a horizontal (or sometime vertical) plane. When the percentage of outcomes is specified at 50%, the corresponding radial error is the familiar circular error probable (CEP). Spatial position error is conveniently characterized by establishing the radius of a sphere that encompasses a prescribed percentage of all possible 3-D position-error outcomes. When the percentage is specified at 50%, the corresponding radial error is the spherical error probable (SEP). The problem of determining CEP and SEP is of long standing and has been the subject of numerous earlier works (see [1–8], for example).

Three approaches have traditionally been taken in determining CEP and SEP. The first approach is to numerically integrate a classic probability integral, thereby providing a tabular definition for the CEP or SEP [2,5,7]. Once the results of the numerical integration are available, they may be used as the basis of a closed-form expression for CEP or SEP. This constitutes the second approach, which has traditionally been used in arriving at approximations for CEP and in which the CEP function is represented over one or more ranges by polynomials chosen to achieve a desired accuracy [8]. This is also the approach taken in the present Note. The third approach is to simplify the probability integral, thereby allowing an approximate closed-form expression for the CEP or SEP [1,4,6]. The best known of the closed-form expressions for CEP and SEP is due to Grubbs [6], with further elaboration and discussion provided in [2,3,8].

This Note deals with the determination of circular and spherical error bounds in covariance-based performance analyses. In this type of analysis, the 3-D statistically distributed position errors are theoretically known via the position-error covariance matrix. It remains only to convert this information into more useful forms, such as CEP or SEP. This Note addresses the question of how this type of conversion may be accurately and conveniently carried out. A set of polynomial functions is numerically derived that accurately characterizes spatial position-error bounds for four distinct cases in which a sphere encompasses 50, 90, 95, and 99% of all possible outcomes. Determination of the corresponding circular error bounds is treated as a special case of the more general problem of determining the spherical error bounds. The accuracies associated with the derived circular and spherical error bounds are defined, and comparisons are made with respect to existing approximations for CEP and SEP.

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Determination of the Principal-Axis Variances of the Position-Error Ellipsoid

The principal axes of the error ellipsoid are generally rotated from the reference frame used in the position-error propagation model. The diagonal elements of the system position-error covariance matrix define the variances of the position errors in the reference frame, with the offdiagonal elements defining the joint expectancies of the three components of position error. In the special case in which the offdiagonal elements of the position-error covariance matrix are zero, the diagonal elements directly define the principal-axis error variances. This is usually not the case, however, and a specialized operation is required to determine the principal-axis error variances. This is accomplished as follows. The position-error covariance matrix P is symmetric and positive semidefinite, which ensures that it can always be decomposed as follows:

$$P = MDM^T \tag{1}$$

where M is an orthogonal transformation matrix for which the columns are the eigenvectors of P, and D is a diagonal matrix for which the elements are the eigenvalues of P. Only the eigenvalues are required, and these define the principal-axis variances of the probability ellipsoid.

For the spatial-error case, the position error is defined by a 3×3 error covariance matrix, the eigenvalues of which are determined by solving the equation

$$\begin{vmatrix} p_{11} - \lambda & p_{12} & p_{13} \\ p_{21} & p_{22} - \lambda & p_{23} \\ p_{31} & p_{32} & p_{33} - \lambda \end{vmatrix} = 0 \tag{2}$$

where p_{ij} are the elements of the position-error covariance matrix. The solution of Eq. (2) leads to the cubic equation

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0 \tag{3}$$

where

$$p = -(p_{11} + p_{22} + p_{33}) \tag{4}$$

$$q = p_{11}p_{22}(1 - r_{12}^2) + p_{11}p_{33}(1 - r_{13}^2) + p_{22}p_{33}(1 - r_{23}^2) \tag{5}$$

$$r = -p_{11}p_{22}p_{33}(1 - r_{12}^2 - r_{13}^2 - r_{23}^2 + 2r_{12}r_{13}r_{23}) \tag{6}$$

and

$$r_{12} = p_{12}/\sqrt{p_{11}p_{22}} \tag{7}$$

$$r_{13} = p_{13}/\sqrt{p_{11}p_{33}} \tag{8}$$

$$r_{23} = p_{23}/\sqrt{p_{22}p_{33}} \tag{9}$$

where r_{ij} are coefficients defining the statistical correlation between the position errors.

The three roots of the cubic equation may be obtained by substituting $x - p/3$ for λ, which results in the reduced cubic equation

$$x^3 + ax + b = 0 \tag{10}$$

where

$$a = \tfrac{1}{3}(3q - p^2) \tag{11}$$

and

$$b = \tfrac{1}{27}(2p^3 - 9pq + 27r) \tag{12}$$

Reformulating Eq. (10) in terms of polar coordinates, the roots of the reduced cubic may be determined as [9]

$$\lambda_1 = m\cos\theta - p/3 \tag{13}$$

$$\lambda_2 = m\cos(\theta + 2\pi/3) - p/3 \tag{14}$$

$$\lambda_3 = m\cos(\theta + 4\pi/3) - p/3 \tag{15}$$

where

$$m = 2\sqrt{\frac{-a}{3}} \tag{16}$$

$$\theta = \frac{1}{3}\cos^{-1}\left[\frac{3b}{am}\right] \tag{17}$$

where λ_i are the variances along the three principal axes of the probability ellipsoid. Because of the positive-definite nature of the position-error covariance matrix, a will always be negative. The arc cosine function can be restricted to the first and second quadrants without loss of generality and the value of its argument can be limited at ± 1 (a possibility only when all three roots of the reduced cubic are equal).

Numerical Solution of the Position-Error Cumulative Probability Integral

The problem of interest is to determine the spherical error bounds that encompass a specified percentage of all possible outcomes for a set of zero-mean, uncorrelated, Gaussian-distributed position errors. In the 3-D spatial problem of interest, the position-error probability density function takes the form of a trivariate Gaussian distribution, and the solution of the following integral equation is required:

$$P(R) = \iiint_{x^2+y^2+z^2 \le R^2} p(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{18}$$

where $P(R)$ is the cumulative probability encompassed by a sphere of radius R centered at the origin of the x–y–z frame, and $p(x,y,z)$ is a zero-mean trivariate Gaussian distribution defined by

$$p(x,y,z) = \frac{\exp\{-[(x/\sigma_x)^2 + (y/\sigma_y)^2 + (z/\sigma_z)^2]/2\}}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} \tag{19}$$

where the standard deviations σ_x, σ_y, and σ_z result from taking the square root of each of the three eigenvalues defined by Eqs. (13–15). The solution of Eq. (18) is expedited by transforming to spherical coordinates; that is, x, y, and z are replaced by

$$x = r\cos\theta\sin\phi \tag{20}$$

$$y = r\sin\theta\sin\phi \tag{21}$$

$$z = r\cos\phi \tag{22}$$

where r is the radial distance measured from the origin of the x–y–z frame, with θ and ϕ being the polar angles. The elemental volume $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ appearing in Eq. (18) is replaced by

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r$$

which allows the integral defining the cumulative probability encompassed by a sphere of radius R to be expressed as

$$P(R) = \int_0^R \int_0^{2\pi} \int_0^{\pi} r^2 p(r,\phi,\theta)\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r \tag{23}$$

where the 3-D probability density function $p(r,\phi,\theta)$ is given by

$$p(r,\phi,\theta) = \frac{\exp\{-(r/\sigma_x)^2[\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta/\alpha^2 + \cos^2\phi/\beta^2]/2\}}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} \tag{24}$$

where

$$\alpha = \sigma_y/\sigma_x \qquad \beta = \sigma_z/\sigma_x$$

Defining

$$r' = r/\sigma_x \qquad R' = R/\sigma_x$$

and recognizing that each octant of the sphere yields the same contribution to the cumulative probability integral allows Eq. (23) to be written in the alternative form:

$$P(R') = 8\int_0^{R'} \int_0^{\pi/2} \int_0^{\pi/2} r'^2 f(r',\phi,\theta)\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r' \tag{25}$$

where

$$f(r',\phi,\theta) = \frac{\exp\{-r'^2[\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta/\alpha^2 + \cos^2\phi/\beta^2]/2\}}{(2\pi)^{3/2}\alpha\beta} \tag{26}$$

Equation (25) does not lend itself to analytical integration and must be evaluated numerically. A 3-D quadrature scheme employing three nested do-loops may be used for this purpose, with the summand being evaluated at the center of each elemental volume. When the desired cumulative probability value is bracketed by two successive iterations of the outer do-loop, the desired value of R' is determined by linear interpolation. The parameters of the elemental volume ensuring an essentially exact value of the integral were established as $\Delta r' = 1/400$ and $\Delta\theta = \Delta\phi = \pi/360$ rad.

Results of applying the numerical integration scheme are shown in Fig. 1 for the four cases that define the radii of spheres encompassing 50, 90, 95, or 99% of all possible random position-error outcomes. The curves define the normalized radius for each value of the standard deviation ratio α ranging from zero to unity for 10 discrete values of the standard deviation ratio β.

Polynomial Approximation to the Spherical Error Bounds

The numerical integration approach for determining the spherical error bounds is very precise but computationally intensive and not convenient for use in system performance analysis. To satisfy the latter objective, it is desirable to use simple functions that approximate the numerical results to an accuracy level that is commensurate with the needs of system performance analysis. A suitable approximation to the results obtained from the numerical integration of Eq. (25) is possible using a polynomial of the form

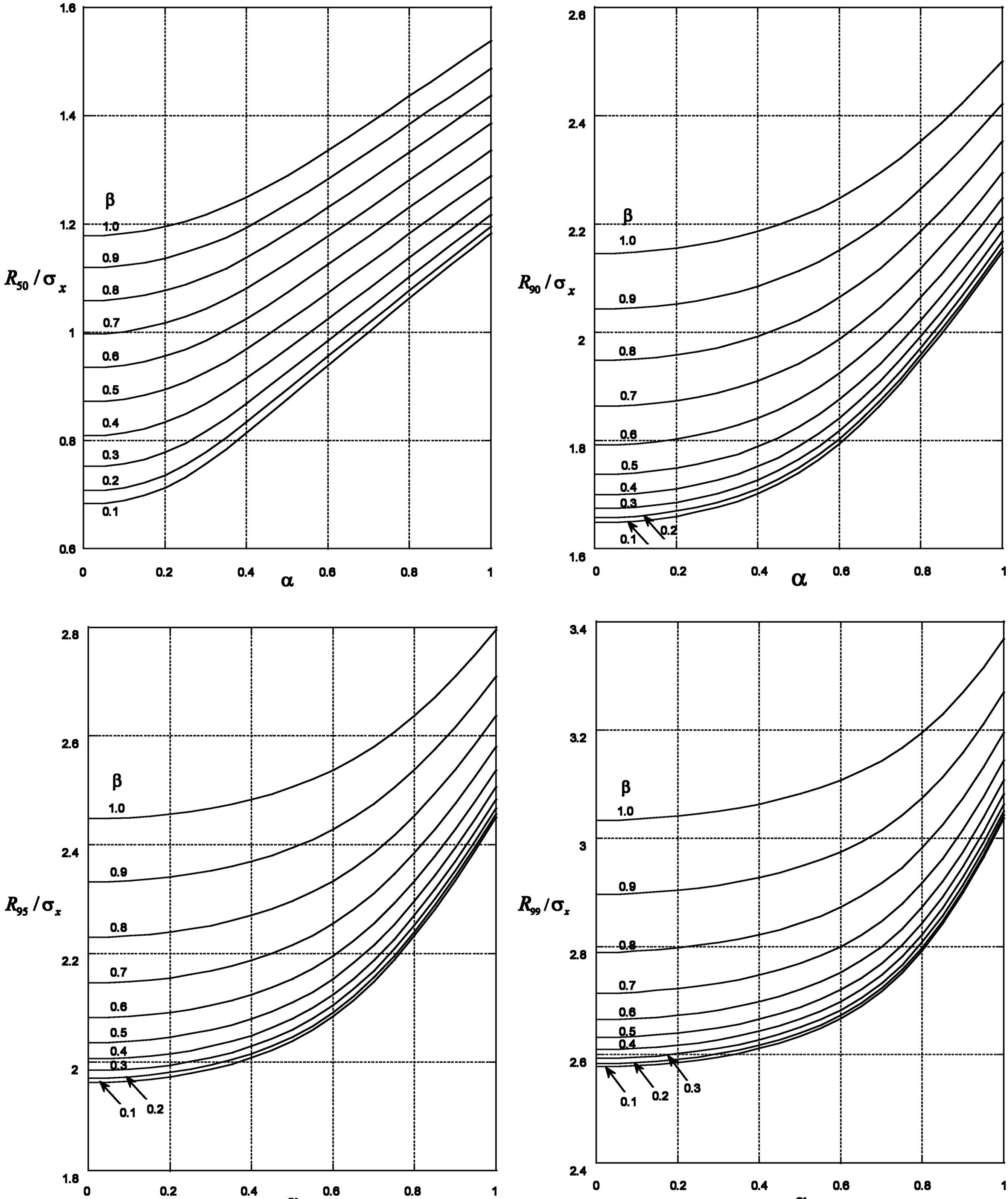

Fig. 1 Spherical error bounds for various cumulative probabilities.

$$
\begin{aligned}
\frac{R_p}{\sigma_x} = {} & (c_{11} + c_{12}\beta + c_{13}\beta^2 + c_{14}\beta^3 + \cdots) \\
& + (c_{21} + c_{22}\beta + c_{23}\beta^2 + c_{24}\beta^3 + \cdots)\alpha \\
& + (c_{31} + c_{32}\beta + c_{33}\beta^2 + c_{34}\beta^3 + \cdots)\alpha^2 \\
& + (c_{41} + c_{42}\beta + c_{43}\beta^2 + c_{44}\beta^3 + \cdots)\alpha^3 + \cdots
\end{aligned}
\tag{27}
$$

where c_{ij} are constants, R_p is the radius of the sphere encompassing $p\%$ of all possible outcomes, and σ_x is the standard deviation along the major axis of the probability ellipsoid, with the parameters α and β as previously defined and, by convention, $\alpha \le 1$ and $\beta \le 1$. The structure of the polynomial function defined by Eq. (27) is predicated upon a basic symmetry property of the numerical results, which is that α and β may be interchanged without effect. It is also clear from this basic property that the convention $\beta \le \alpha \le 1$ may be imposed without loss of generality.

The form defined by Eq. (27) was used to fit the computed data via a least-squares solution incorporating a triangular array of 820 computed points for α and β at intervals of 0.025. By experimenting with polynomials of various orders, it was found that the approximation errors associated with a third-order polynomial in α and β for each of the four families considered are at a level commensurate with the needs of system performance analysis. The coefficients thus derived are given in Table 1 for the four families of spherical error bounds. The approximating polynomials defined by the Table 1 coefficients are also applicable in computing the circular error bounds for the 2-D case, by setting β to zero. This first requires that the position-error variances along the two principal axes of the probability ellipse be determined from the 2×2 equivalent of Eq. (2).

A well-known alternate closed-form expression for SEP, due to Grubbs [6], is given by

$$\mathrm{SEP}/\sigma_x = [(1+\alpha^2+\beta^2)(1-V/9)^3]^{1/2} \tag{28}$$

with V defined as

$$V = 2\frac{(1+\alpha^4+\beta^4)}{(1+\alpha^2+\beta^2)^2} \tag{29}$$

where σ_x is the largest of the standard deviations, and α and β are the ratios defined earlier. The Grubbs formulation also allows the CEP to be computed simply by setting $\beta = 0$ in Eqs. (28) and (29).

The errors associated with Eq. (27) may be determined by comparing the essentially exact values determined by numerical integration of Eq. (25) with the approximate values. The results are shown in Tables 2 and 3 for the 50, 90, 95, and 99% cases for both the spherical and circular error bounds. For the 50% cases, the errors associated with the Grubbs approximation are also provided for comparison.

It is clear that the expression defined for computing the circular and spherical error bounds provides accuracies very compatible with the needs of contemporary covariance-based linear performance analysis, which is typically limited more significantly by the accuracy and fidelity of the position-error propagation model used. Furthermore, as is evident from Tables 2 and 3, the accuracies associated with the 50% circular and spherical error bounds are better than those associated with the Grubbs approximation. Since the Grubbs approximation is not applicable to the 90, 95, and 99% cases, no comparisons are possible for these cases. Still another comparison can be made for an existing closed-form CEP expression, as defined in [1], which states that errors as large as 2% are possible in the derived CEP, depending on the correlation coefficient for the two components of position error.

Table 1 Polynomial coefficients for spherical position-error bounds

Coefficients[a]	R_{50}	R_{90}	R_{95}	R_{99}
c_{11}	0.6754	1.6494	1.9626	2.5686
c_{12}	−0.1547	0.0332	−0.0906	−0.1150
c_{13}	0.2616	1.3376	1.3214	−0.3475
c_{14}	1.0489	−0.8445	−0.3994	1.3570
c_{21}	−0.0208	−0.0588	0.0100	0.1479
c_{22}	1.1739	−0.5605	0.2722	0.9950
c_{23}	1.9540	−4.7170	−5.4821	1.3223
c_{24}	−5.5678	5.7135	3.9732	−4.8917
c_{31}	1.1009	0.3996	0.0700	−0.4285
c_{32}	−2.6375	1.5739	0.0462	−1.9795
c_{33}	−1.4838	5.3623	7.1658	−1.1104
c_{34}	6.5837	−7.9347	−5.7194	6.9617
c_{41}	−0.5821	0.1636	0.4092	0.7371
c_{42}	1.5856	−1.0747	−0.1953	1.2495
c_{43}	−0.0678	−1.7785	−3.0134	−0.2061
c_{44}	−2.3324	3.2388	2.4661	−2.8968

[a]Coefficients in a polynomial of the form $R_p/\sigma_x = c_{11} + c_{12}\beta + c_{13}\beta^2 + c_{14}\beta^3 + (c_{21} + c_{22}\beta + c_{23}\beta^2 + c_{24}\beta^3)\alpha + (c_{31} + c_{32}\beta + c_{33}\beta^2 + c_{34}\beta^3)\alpha^2 + (c_{41} + c_{42}\beta + c_{43}\beta^2 + c_{44}\beta^3)\alpha^3$, where $\alpha = \sigma_y/\sigma_x$ and $\beta = \sigma_z/\sigma_x$ and where, by convention, $\beta \le \alpha \le 1$.

Table 2 Maximum percentage errors associated with the computed spherical position-error bounds

Range[a]	R_{50}[b]	R_{90}	R_{95}	R_{99}
0.025–0.25	1.0 (2.7)	0.3	0.2	0.2
0.25–0.50	0.5 (2.4)	0.2	0.2	0.1
0.50–0.75	0.4 (1.0)	0.2	0.2	0.1
0.75–1.0	0.5 (1.0)	0.3	0.2	0.1

[a]Range of values for α, with $\beta \le \alpha$.
[b]Values in parentheses indicate corresponding maximum percent error using the Grubbs formula for SEP.

Table 3 Maximum percentage errors associated with the computed circular position-error bounds

Range[a]	R_{50}[b]	R_{90}	R_{95}	R_{99}
0.025–0.25	0.7 (2.3)	0.2	0.2	0.1
0.25–0.50	0.4 (2.0)	0.2	0.1	0.2
0.50–0.75	0.6 (0.5)	0.2	0.2	0.2
0.75–1.0	0.7 (0.7)	0.3	0.3	0.3

[a]Range of values for α.
[b]Values in parentheses indicate corresponding maximum percent error using the Grubbs formula for CEP.

Conclusions

This Note has defined a simple and accurate approach for determining families of circular and spherical error bounds that characterize random position errors. Sets of third-order polynomial functions were defined for the 50, 90, 95, and 99% circular and spherical position-error bounds. Although simple in structure, the polynomial functions yield accuracies very compatible with the needs of contemporary covariance-based system performance analyses. Furthermore, the numerical accuracy associated with the computation of SEP and CEP was shown to be better than that of the commonly used Grubbs formula.

It may be concluded that a relatively simple polynomial function, together with a set of 64 constants, can provide accurate values for all circular and spherical position-error bounds typically required in covariance-based performance analysis. The technique is also applicable to the accurate determination of still other members of the families of circular and spherical position-error bounds not considered here, with the form given by Eq. (27) being applicable as well. Higher-order realizations of Eq. (27) are also possible if greater accuracy is desired.

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